

Common properties of bounded linear operators AC and BA : Spectral theory ^{*}

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Abstract: Let X, Y be Banach spaces, $A : X \longrightarrow Y$ and $B, C : Y \longrightarrow X$ be bounded linear operators satisfying operator equation $ABA = ACA$. Recently, as extensions of Jacobson's lemma, Corach, Duggal and Harte studied common properties of $AC - I$ and $BA - I$ in algebraic viewpoint and also obtained some topological analogues. In this note, we continue to investigate common properties of AC and BA from the viewpoint of spectral theory. In particular, we give an affirmative answer to one question posed by Corach et al. by proving that $AC - I$ has closed range if and only if $BA - I$ has closed range.

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1 Introduction and Notations

For any Banach spaces X and Y , let $\mathcal{B}(X, Y)$ denote the set of all bounded linear operators from X to Y . Jacobson's lemma [1, 2, 3, 7, 10, 12] states that if $A \in \mathcal{B}(X, Y)$ and $C \in \mathcal{B}(Y, X)$ then

$$AC - I \text{ is invertible} \iff CA - I \text{ is invertible.} \quad (1.1)$$

Recently, we generalized (1.1) to various regularities, in the sense of Kordula and Müller, and showed that $AC - I$ and $CA - I$ share common complementability of kernels (see [13]). In 2013, Corach, Duggal and Harte [5] extended (1.1) and many of its relatives form $CA - I$ to certain $BA - I$ under the assumption

$$ABA = ACA, \quad (1.2)$$

where $A \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$. In this note, we continue to study this situation and show that AC and BA share many common spectral properties. For

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example, answering one question posed by Corach et al. in [5, p. 526], we prove that $AC - I$ has closed range if and only if $BA - I$ has closed range. But at present we are unable to decide whether $AC - I$ and $BA - I$ share common complementability of kernels. For some other open questions in this direction, we refer the reader to [5, 13].

We first fix some notations in spectral theory. Throughout this paper, $\mathcal{B}(X) = \mathcal{B}(X, X)$. For an operator $T \in \mathcal{B}(X)$, let $\mathcal{N}(T)$ denote its kernel, $\mathcal{R}(T)$ its range and $\sigma(T)$ its spectrum. For each $n \in \mathbb{N} := \{0, 1, 2, \dots\}$, we set $c_n(T) = \dim \mathcal{R}(T^n)/\mathcal{R}(T^{n+1})$ and $c'_n(T) = \dim \mathcal{N}(T^{n+1})/\mathcal{N}(T^n)$. It is well known that ([8, Lemmas 3.2 and 3.1]), for every $n \in \mathbb{N}$,

$$c_n(T) = \dim X/(\mathcal{R}(T) + \mathcal{N}(T^n)), \quad c'_n(T) = \dim \mathcal{N}(T) \cap \mathcal{R}(T^n).$$

Hence, it is easy to see that the sequences $\{c_n(T)\}_{n=0}^\infty$ and $\{c'_n(T)\}_{n=0}^\infty$ are decreasing. For each $n \in \mathbb{N}$, T induces a linear transformation from the vector space $\mathcal{R}(T^n)/\mathcal{R}(T^{n+1})$ to the space $\mathcal{R}(T^{n+1})/\mathcal{R}(T^{n+2})$. We will let $k_n(T)$ be the dimension of the null space of the induced map and let

$$k(T) = \sum_{n=0}^{\infty} k_n(T).$$

From Lemma 2.3 in [6] it follows that, for every $n \in \mathbb{N}$,

$$\begin{aligned} k_n(T) &= \dim(\mathcal{N}(T) \cap \mathcal{R}(T^n))/(\mathcal{N}(T) \cap \mathcal{R}(T^{n+1})) \\ &= \dim(\mathcal{R}(T) + \mathcal{N}(T^{n+1}))/(\mathcal{R}(T) + \mathcal{N}(T^n)). \end{aligned}$$

We remark that the sequence $\{k_n(T)\}_{n=0}^\infty$ is not always decreasing. From Theorem 3.7 in [6] it follows that

$$k(T) = \dim \mathcal{N}(T)/(\mathcal{N}(T) \cap \mathcal{R}(T^\infty)) = \dim(\mathcal{R}(T) + \mathcal{N}(T^\infty))/\mathcal{R}(T).$$

Just as the definition of $k(T)$, we give the definitions of *stable nullity* $c'(T)$ and *stable defect* $c(T)$ as follows.

Definition 1.1. *Let $T \in \mathcal{B}(X)$. The stable nullity $c'(T)$ of T is defined as*

$$c'(T) = \sum_{n=0}^{\infty} c'_n(T),$$

and the stable defect $c(T)$ of T is defined as

$$c(T) = \sum_{n=0}^{\infty} c_n(T).$$

It is easy to see that $c(T) = \dim X/\mathcal{R}(T^\infty)$ and $c'(T) = \dim \mathcal{N}(T^\infty)$.

In [9], Kordula and Müller defined the concept of *regularity* as follows:

Definition 1.2. ([9]) *A non-empty subset $\mathbf{R} \subseteq \mathcal{B}(X)$ is called a regularity if it satisfies the following conditions:*

- (1) *If $A \in \mathcal{B}(X)$ and $n \geq 1$, then $A \in \mathbf{R}$ if and only if $A^n \in \mathbf{R}$.*
- (2) *If $A, B, C, D \in \mathcal{B}(X)$ are mutually commuting operators satisfying $AC + BD = I$, then $AB \in \mathbf{R}$ if and only if $A, B \in \mathbf{R}$.*

A non-empty subset $\mathbf{R} \subseteq \mathcal{B}(X)$ defines in a natural way a spectrum by $\sigma_{\mathbf{R}}(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathbf{R}\}$, for every $T \in \mathcal{B}(X)$. The crucial property of the spectrum $\sigma_{\mathbf{R}}$ corresponding to a regularity \mathbf{R} is that it satisfies a restricted spectral mapping theorem ([9, Theorem 1.4]),

$$f(\sigma_{\mathbf{R}}(T)) = \sigma_{\mathbf{R}}(f(T))$$

for every function f analytic on a neighbourhood of $\sigma(T)$ which is non-constant on each component of its domain of definition.

We now give the definitions of some concrete subsets $\mathbf{R}_i \subseteq \mathcal{B}(X)$, $1 \leq i \leq 19$. For the fact that \mathbf{R}_i ($1 \leq i \leq 19$) form a regularity, the reader should refer to [4, 11, 13].

Definition 1.3. $\mathbf{R}_1 = \{T \in \mathcal{B}(X) : c(T) = 0\}$,

$$\mathbf{R}_2 = \{T \in \mathcal{B}(X) : c(T) < \infty\},$$

$$\mathbf{R}_3 = \{T \in \mathcal{B}(X) : \text{there exists } d \in \mathbb{N} \text{ such that } c_d(T) = 0 \text{ and } \mathcal{R}(T^{d+1}) \text{ is closed}\},$$

$$\mathbf{R}_4 = \{T \in \mathcal{B}(X) : c_n(T) < \infty \text{ for every } n \in \mathbb{N}\},$$

$$\mathbf{R}_5 = \{T \in \mathcal{B}(X) : \text{there exists } d \in \mathbb{N} \text{ such that } c_d(T) < \infty \text{ and } \mathcal{R}(T^{d+1}) \text{ is closed}\},$$

$$\mathbf{R}_6 = \{T \in \mathcal{B}(X) : c'(T) = 0 \text{ and } \mathcal{R}(T) \text{ is closed}\},$$

$$\mathbf{R}_7 = \{T \in \mathcal{B}(X) : c'(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed}\},$$

$$\mathbf{R}_8 = \{T \in \mathcal{B}(X) : \text{there exists } d \in \mathbb{N} \text{ such that } c'_d(T) = 0 \text{ and } \mathcal{R}(T^{d+1}) \text{ is closed}\},$$

$$\mathbf{R}_9 = \{T \in \mathcal{B}(X) : c'_n(T) < \infty \text{ for every } n \in \mathbb{N} \text{ and } \mathcal{R}(T) \text{ is closed}\},$$

$$\mathbf{R}_{10} = \{T \in \mathcal{B}(X) : \text{there exists } d \in \mathbb{N} \text{ such that } c'_d(T) < \infty \text{ and } \mathcal{R}(T^{d+1}) \text{ is closed}\},$$

$$\mathbf{R}_{11} = \{T \in \mathcal{B}(X) : k(T) = 0 \text{ and } \mathcal{R}(T) \text{ is closed}\},$$

$$\mathbf{R}_{12} = \{T \in \mathcal{B}(X) : k(T) < \infty \text{ and } \mathcal{R}(T) \text{ is closed}\},$$

$$\mathbf{R}_{13} = \{T \in \mathcal{B}(X) : \text{there exists } d \in \mathbb{N} \text{ such that } k_n(T) = 0 \text{ for every } n \geq d \text{ and } \mathcal{R}(T^{d+1}) \text{ is closed}\},$$

$$\mathbf{R}_{14} = \{T \in \mathcal{B}(X) : k_n(T) < \infty \text{ for every } n \in \mathbb{N} \text{ and } \mathcal{R}(T) \text{ is closed}\},$$

$$\mathbf{R}_{15} = \{T \in \mathcal{B}(X) : \text{there exists } d \in \mathbb{N} \text{ such that } k_n(T) < \infty \text{ for every } n \geq d \text{ and } \mathcal{R}(T^{d+1}) \text{ is closed}\}.$$

$$\mathbf{R}_{16} = \{T \in \mathcal{B}(X) : \text{there exists } d \in \mathbb{N} \text{ such that } c_d(T) = 0 \text{ and } \mathcal{R}(T) + \mathcal{N}(T^d) \text{ is closed}\},$$

$$\mathbf{R}_{17} = \{T \in \mathcal{B}(X) : \text{there exists } d \in \mathbb{N} \text{ such that } c_d(T) < \infty \text{ and } \mathcal{R}(T) + \mathcal{N}(T^d) \text{ is closed}\},$$

$$\mathbf{R}_{18} = \{T \in \mathcal{B}(X) : \text{there exists } d \in \mathbb{N} \text{ such that } k_n(T) = 0 \text{ for every } n \geq d, \text{ and } \mathcal{R}(T) + \mathcal{N}(T^d) \text{ is closed}\},$$

$$\mathbf{R}_{19} = \{T \in \mathcal{B}(X) : \text{there exists } d \in \mathbb{N} \text{ such that } k_n(T) < \infty \text{ for every } n \geq d, \text{ and } \mathcal{R}(T) + \mathcal{N}(T^d) \text{ is closed}\}.$$

We remark that $\mathbf{R}_1 \subseteq \mathbf{R}_2 = \mathbf{R}_3 \cap \mathbf{R}_4 \subseteq \mathbf{R}_3 \cup \mathbf{R}_4 \subseteq \mathbf{R}_5 \subseteq \mathbf{R}_{13} \subseteq \mathbf{R}_{18}$, $\mathbf{R}_3 \subseteq \mathbf{R}_{16}$, $\mathbf{R}_5 \subseteq \mathbf{R}_{17}$, $\mathbf{R}_6 \subseteq \mathbf{R}_7 = \mathbf{R}_8 \cap \mathbf{R}_9 \subseteq \mathbf{R}_8 \cup \mathbf{R}_9 \subseteq \mathbf{R}_{10} \subseteq \mathbf{R}_{13}$, $\mathbf{R}_{11} \subseteq \mathbf{R}_{12} = \mathbf{R}_{13} \cap \mathbf{R}_{14} \subseteq \mathbf{R}_{13} \cup \mathbf{R}_{14} \subseteq \mathbf{R}_{15} \subseteq \mathbf{R}_{19}$. The operators of \mathbf{R}_1 , \mathbf{R}_2 , \mathbf{R}_3 , \mathbf{R}_4 and \mathbf{R}_5 are called surjective, lower semi-Browder, right Drazin invertible, lower semi-Fredholm and right essentially Drazin invertible operators, respectively. The operators of \mathbf{R}_6 , \mathbf{R}_7 , \mathbf{R}_8 , \mathbf{R}_9 and \mathbf{R}_{10} are called bounded below, upper semi-Browder, left Drazin invertible, upper semi-Fredholm and left essentially Drazin invertible operators, respectively. The operators of \mathbf{R}_{11} , \mathbf{R}_{12} and \mathbf{R}_{13} are called semi-regular, essentially semi-regular and quasi-Fredholm operators,

respectively. The operators of \mathbf{R}_{18} are called operators with eventual topological uniform descent.

The main result of this note establishes that if $A \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy (1.2), then

$$AC - I \in \mathbf{R}_i \iff BA - I \in \mathbf{R}_i, \quad 1 \leq i \leq 19.$$

It not only extends our previous corresponding results in [13] from the special case

$$B = C$$

to the general case, but also supplements the results obtained by Corach, Duggal and Harte in [5] from the viewpoint of spectral theory.

2 Main result

Throughout this section, we assume that $A \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy (1.2). We begin with the following lemma, which gives an affirmative answer to one question posed by Corach et al. in [5, p. 526].

Lemma 2.1. *$\mathcal{R}(AC - I)$ is closed if and only if $\mathcal{R}(BA - I)$ is closed.*

Proof. We give the proof by taking the argument in [5, pp. 526-527] one step further. Suppose that $\mathcal{R}(AC - I)$ is closed. Assume that there exists a sequence $\{x_n\}_{n=1}^\infty \subseteq \mathcal{R}(BA - I)$ such that $x_n \rightarrow x$. Then, for each positive integer n , there exists $z_n \in X$ such that $x_n = (BA - I)z_n$. Hence

$$Ax = \lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} A(BA - I)z_n = \lim_{n \rightarrow \infty} (AC - I)Az_n.$$

Since $\mathcal{R}(AC - I)$ is closed, there exists $y \in Y$ such that

$$Ax = (AC - I)y,$$

and hence $y = ACy - Ax$. Therefore,

$$\begin{aligned} x &= BAx - (BA - I)x \\ &= B(AC - I)y - (BA - I)x \\ &= (BAC - B)(ACy - Ax) - (BA - I)x \\ &= BACACy - BACAx - BACy + BAx - (BA - I)x \\ &= BABACy - BABAx - BACy + BAx - (BA - I)x \\ &= (BA - I)(BACy - BAx - x), \end{aligned}$$

that is, $x \in \mathcal{R}(BA - I)$. Consequently, $\mathcal{R}(BA - I)$ is closed.

Conversely, suppose that $\mathcal{R}(BA - I)$ is closed. Then we have

$$\begin{aligned} \mathcal{R}(BA - I) \text{ is closed} &\iff \mathcal{R}(AB - I) \text{ is closed} \quad (\text{by [2, Theorem 5]}) \\ &\implies \mathcal{R}(CA - I) \text{ is closed} \\ &\quad (\text{by the previous paragraph and interchanging } B \text{ and } C) \\ &\iff \mathcal{R}(AC - I) \text{ is closed.} \quad (\text{by [2, Theorem 5]}) \end{aligned}$$

Hence, $\mathcal{R}(AC - I)$ is closed $\iff \mathcal{R}(BA - I)$ is closed. \square

Lemma 2.2. *For all $n \in \mathbb{N}$, $\mathcal{R}((AC - I)^n)$ is closed if and only if $\mathcal{R}((BA - I)^n)$ is closed.*

Proof. For each $n \in \mathbb{N}$, let

$$B_n = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} B(AB)^{k-1}$$

and

$$C_n = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} C(AC)^{k-1}.$$

Then we have

$$\begin{aligned} AB_n A &= AC_n A, \\ (I - AC)^{n+1} &= I - AC_n \end{aligned}$$

and

$$(I - BA)^{n+1} = I - B_n A.$$

Applying Lemma 2.1, we get

$$\mathcal{R}((AC - I)^n) \text{ is closed} \iff \mathcal{R}((BA - I)^n) \text{ is closed},$$

for all $n \in \mathbb{N}$. \square

The next lemma is essential to the sequel crucial lemmas 2.4–2.7.

Lemma 2.3. *For all $d \in \mathbb{N}$, we have*

- (1) $A\mathcal{R}((BA - I)^d) \subseteq \mathcal{R}((AC - I)^d)$;
- (2) $A\mathcal{N}((BA - I)^d) \subseteq \mathcal{N}((AC - I)^d)$;
- (3) $BAC\mathcal{N}((AC - I)^d) \subseteq \mathcal{N}((BA - I)^d)$;
- (4) $BAC\mathcal{R}((AC - I)^d) \subseteq \mathcal{R}((BA - I)^d)$.

Proof. (1) Let $x \in \mathcal{R}((BA - I)^d)$. Then there exists $x_0 \in X$ such that $x = (BA - I)^d x_0$, hence

$$Ax = A(BA - I)^d x_0 = (AC - I)^d Ax_0 \in \mathcal{R}((AC - I)^d).$$

Therefore, $A\mathcal{R}((BA - I)^d) \subseteq \mathcal{R}((AC - I)^d)$.

- (2) Let $x \in \mathcal{N}((BA - I)^d)$. Then we have

$$(AC - I)^d Ax = A(BA - I)^d x = 0.$$

Thus $Ax \in \mathcal{N}((AC - I)^d)$ and this shows (2).

- (3) Let $y \in \mathcal{N}((AC - I)^d)$. Then we have

$$(BA - I)^d BACy = BAC(AC - I)^d y = 0.$$

Thus $BACy \in \mathcal{N}((BA - I)^d)$ and this shows (3).

(4) Let $y \in \mathcal{R}((AC - I)^d)$. Then there exists $y_0 \in Y$ such that $y = (AC - I)^d y_0$, hence

$$BACy = BAC(AC - I)^d y_0 = (BA - I)^d BACy_0 \in \mathcal{R}((BA - I)^d).$$

Therefore, $BAC\mathcal{R}((AC - I)^d) \subseteq \mathcal{R}((BA - I)^d)$. \square

The proofs of the following lemmas 2.4–2.7 are dependent heavily on the special case

$$B = C,$$

which we proved recently in [13].

Lemma 2.4. *For all $d \in \mathbb{N}$, $\mathcal{R}(AC - I) + \mathcal{N}((AC - I)^d)$ is closed if and only if $\mathcal{R}(BA - I) + \mathcal{N}((BA - I)^d)$ is closed.*

Proof. Suppose that $\mathcal{R}(AC - I) + \mathcal{N}((AC - I)^d)$ is closed. Assume that there exists a sequence $\{x_n\}_{n=1}^\infty \subseteq \mathcal{R}(BA - I) + \mathcal{N}((BA - I)^d)$ such that $x_n \rightarrow x$. Then, for each positive integer n , there exist $y_n \in \mathcal{R}(BA - I)$ and $z_n \in \mathcal{N}((BA - I)^d)$ such that $x_n = y_n + z_n$. Hence

$$Ax = \lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} A(y_n + z_n).$$

Parts (1) and (2) of Lemma 2.3 imply that

$$Ay_n \in \mathcal{R}(AC - I)$$

and

$$Az_n \in \mathcal{N}((AC - I)^d).$$

Since $\mathcal{R}(AC - I) + \mathcal{N}((AC - I)^d)$ is closed, there exist $y \in Y$ and $z \in \mathcal{N}((AC - I)^d)$ such that

$$Ax = (AC - I)y + z,$$

and hence $y = ACy + z - Ax$. Therefore,

$$\begin{aligned} x &= BAx - (BA - I)x \\ &= B[(AC - I)y + z] - (BA - I)x \\ &= (BAC - B)(ACy + z - Ax) + Bz - (BA - I)x \\ &= BACACy + BACz - BACAx - BACy - Bz + BAx + Bz - (BA - I)x \\ &= BABACy + BACz - BABAx - BACy + BAx - (BA - I)x \\ &= (BA - I)(BACy - BAx - x) + BACz. \end{aligned}$$

And then, since $z \in \mathcal{N}((AC - I)^d)$, we get $x \in \mathcal{R}(BA - I) + \mathcal{N}((BA - I)^d)$ by Lemma 2.3(3). Consequently, $\mathcal{R}(BA - I) + \mathcal{N}((BA - I)^d)$ is closed.

Conversely, suppose that $\mathcal{R}(BA - I) + \mathcal{N}((BA - I)^d)$ is closed. Then we have

$$\begin{aligned} &\mathcal{R}(BA - I) + \mathcal{N}((BA - I)^d) \text{ is closed} \\ \iff &\mathcal{R}(AB - I) + \mathcal{N}((AB - I)^d) \text{ is closed} \quad (\text{by [13, Lemma 3.11]}) \\ \implies &\mathcal{R}(CA - I) + \mathcal{N}((CA - I)^d) \text{ is closed} \\ &(\text{by the previous paragraph and interchanging } B \text{ and } C) \\ \iff &\mathcal{R}(AC - I) + \mathcal{N}((AC - I)^d) \text{ is closed.} \quad (\text{by [13, Lemma 3.11]}) \end{aligned}$$

Hence, $\mathcal{R}(AC - I) + \mathcal{N}((AC - I)^d)$ is closed $\iff \mathcal{R}(BA - I) + \mathcal{N}((BA - I)^d)$ is closed. \square

Lemma 2.5. $c'_n(AC - I) = c'_n(BA - I)$ for all $n \in \mathbb{N}$. Consequently, $c'(AC - I) = c'(BA - I)$.

Proof. Let $\widehat{A}(c'_n)$ be the linear mapping induced by A from

$$\mathcal{N}((BA - I)^{n+1})/\mathcal{N}((BA - I)^n)$$

to

$$\mathcal{N}((AC - I)^{n+1})/\mathcal{N}((AC - I)^n).$$

Since $A\mathcal{N}((BA - I)^n) \subseteq \mathcal{N}((AC - I)^n)$ (see Lemma 2.3(2)), we thus know that $\widehat{A}(c'_n)$ is well defined.

Next, we show that $\widehat{A}(c'_n)$ is injective. In fact, let $x \in \mathcal{N}((BA - I)^{n+1})$ and $Ax \in \mathcal{N}((AC - I)^n)$. Then by Lemma 2.3(3), we have $BACAx \in \mathcal{N}((BA - I)^n)$. Hence,

$$\begin{aligned} x &= BAx - (BA - I)x \\ &= BAx - BACAx + BACAx - (BA - I)x \\ &= BAx - BABAx + BACAx - (BA - I)x \\ &= BA(I - BA)x + BACAx - (BA - I)x \\ &\in \mathcal{N}((BA - I)^n). \end{aligned}$$

Therefore, $\widehat{A}(c'_n)$ is injective.

Hence,

$$\begin{aligned} c'_n(BA - I) &\leq c'_n(AC - I) \quad (\text{by the previous paragraph}) \\ &= c'_n(CA - I) \quad (\text{by [13, Lemma 3.10]}) \\ &\leq c'_n(AB - I) \\ &\quad (\text{by the previous paragraph and interchanging } B \text{ and } C) \\ &= c'_n(BA - I) \quad (\text{by [13, Lemma 3.10]}) \end{aligned}$$

So $c'_n(AC - I) = c'_n(BA - I)$ for all $n \in \mathbb{N}$. \square

Lemma 2.6. $c_n(AC - I) = c_n(BA - I)$ for all $n \in \mathbb{N}$. Consequently, $c(AC - I) = c(BA - I)$.

Proof. Let $\widehat{A}(c_n)$ be the linear mapping induced by A from

$$X/(\mathcal{R}(BA - I) + \mathcal{N}((BA - I)^n))$$

to

$$Y/(\mathcal{R}(AC - I) + \mathcal{N}((AC - I)^n)).$$

Since

$$A(\mathcal{R}(BA - I) + \mathcal{N}((BA - I)^n)) \subseteq \mathcal{R}(AC - I) + \mathcal{N}((AC - I)^n)$$

(by Lemma 2.3(1) and (2)), we thus know that $\widehat{A}(c'_n)$ is well defined.

Next, we show that $\widehat{A}(c_n)$ is injective. In fact, let $x \in X$ and $Ax \in \mathcal{R}(AC - I) + \mathcal{N}((AC - I)^n)$. Then by the proof of the first paragraph in Lemma 2.4, we get $x \in \mathcal{R}(BA - I) + \mathcal{N}((BA - I)^n)$. Therefore, $\widehat{A}(c'_n)$ is injective.

Hence,

$$\begin{aligned} c_n(BA - I) &\leq c_n(AC - I) \quad (\text{by the previous paragraph}) \\ &= c_n(CA - I) \quad (\text{by [13, Lemma 3.9]}) \\ &\leq c_n(AB - I) \\ &\quad (\text{by the previous paragraph and interchanging } B \text{ and } C) \\ &= c_n(BA - I) \quad (\text{by [13, Lemma 3.9]}) \end{aligned}$$

So $c_n(AC - I) = c_n(BA - I)$ for all $n \in \mathbb{N}$. \square

Lemma 2.7. $k_n(AC - I) = k_n(BA - I)$ for all $n \in \mathbb{N}$. Consequently, $k(AC - I) = k(BA - I)$.

Proof. Let $\widehat{A}(k_n)$ be the linear mapping induced by A from

$$(\mathcal{N}(BA - I) \cap \mathcal{R}((BA - I)^n)) / (\mathcal{N}(BA - I) \cap \mathcal{R}((BA - I)^{n+1}))$$

to

$$(\mathcal{N}(AC - I) \cap \mathcal{R}((AC - I)^n)) / (\mathcal{N}(AC - I) \cap \mathcal{R}((AC - I)^{n+1})).$$

Since

$$A(\mathcal{N}(BA - I) \cap \mathcal{R}((BA - I)^{n+1})) \subseteq \mathcal{N}(AC - I) \cap \mathcal{R}((AC - I)^{n+1})$$

(by Lemma 2.3(1) and (2)), we thus know that $\widehat{A}(k_n)$ is well defined.

Next, we show that $\widehat{A}(k_n)$ is injective. In fact, let $x \in \mathcal{N}(BA - I) \cap \mathcal{R}((BA - I)^n)$ and $Ax \in \mathcal{N}(AC - I) \cap \mathcal{R}((AC - I)^{n+1})$. Then by Lemma 2.3(4), we have $BACAx \in \mathcal{R}((BA - I)^{n+1})$. Hence,

$$\begin{aligned} x &= BAx - (BA - I)x \\ &= BAx - BACAx + BACAx - (BA - I)x \\ &= BAx - BABAx + BACAx - (BA - I)x \\ &= BA(I - BA)x + BACAx - (BA - I)x \\ &\in \mathcal{R}((BA - I)^{n+1}), \end{aligned}$$

thus $x \in \mathcal{N}(BA - I) \cap \mathcal{R}((BA - I)^{n+1})$. Therefore, $\widehat{A}(c'_n)$ is injective.

Hence,

$$\begin{aligned} k_n(BA - I) &\leq k_n(AC - I) \quad (\text{by the previous paragraph}) \\ &= k_n(CA - I) \quad (\text{by [13, Lemma 3.8]}) \\ &\leq k_n(AB - I) \\ &\quad (\text{by the previous paragraph and interchanging } B \text{ and } C) \\ &= k_n(BA - I) \quad (\text{by [13, Lemma 3.8]}) \end{aligned}$$

So $k_n(AC - I) = k_n(BA - I)$ for all $n \in \mathbb{N}$. \square

Since the basic components of regularities \mathbf{R}_i ($1 \leq i \leq 19$) are all considered in Lemmas 2.1, 2.2 and 2.4–2.7, we are now in a position to give the proof of the following main result.

Theorem 2.8. $\sigma_{\mathbf{R}_i}(AC) \setminus \{0\} = \sigma_{\mathbf{R}_i}(BA) \setminus \{0\}$ for $1 \leq i \leq 19$.

Proof. Applying Lemmas 2.1, 2.2 and 2.4–2.7, the desired conclusion follows directly. \square

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